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New bounds on the price of bandit feedback for mistake-bounded online multiclass learning

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Abstract

This paper is about two generalizations of the mistake bound model to online multiclass classification. In the *standard model*, the learner receives the correct classification at the end of each round, and in the *bandit model*, the learner only finds out whether its prediction was correct or not. For a set F of multiclass classifiers, let $\text{opt}_{\text{std}}(F)$ and $\text{opt}_{\text{bandit}}(F)$ be the optimal bounds for learning F according to these two models. We show that an

$$\text{opt}_{\text{bandit}}(F) \leq (1 + o(1))(|Y| \ln |Y|)\text{opt}_{\text{std}}(F)$$

bound is the best possible up to the leading constant, closing a $\Theta(\log |Y|)$ factor gap.
Keywords: Mistake bounds, multiclass classification, bandit feedback, complexity.

1. Introduction

There are two natural ways to generalize the mistake-bound model (Littlestone, 1988) to multiclass classification (Auer et al., 1995).

In the *standard model*, for a set F of functions from some set X to a finite set Y, for an arbitrary $f \in F$ that is unknown to the algorithm, learning proceeds in rounds, and in round t, the algorithm

- receives $x_t \in X$,
- predicts $\hat{y}_t \in Y$, and
- gets $f(x_t)$.

The goal is to bound the number of prediction mistakes in the worst case, over all possible $f \in F$ and $x_1, x_2, ... \in X$.

The *bandit model* (Dani et al., 2008; Crammer and Gentile, 2013; Hazan and Kale, 2011) (called "weak reinforcement" in (Auer et al., 1995; Auer and Long, 1999)) is like the standard model, except that, at the end of each round, the algorithm only finds out whether $\hat{y}_t = f(x_t)$ or not.

Obviously, $\text{opt}_{\text{std}}(F) \leq \text{opt}_{\text{bandit}}(F)$. It is known (Auer and Long, 1999) that, for all F,

$$\text{opt}_{\text{bandit}}(F) \leq (2.01 + o(1))\, (|Y| \ln |Y|)\, \text{opt}_{\text{std}}(F), \tag{1}$$

and that, for any k and M, there is a set F of functions from a set X to a set Y of size k such that $\mathrm{opt}_{\mathrm{std}}(F) = M$ and

$$\mathrm{opt}_{\mathrm{bandit}}(F) \geq (|Y| - 1)\,\mathrm{opt}_{\mathrm{std}}(F),$$

so that (1) cannot be improved by more than a log factor.

This note shows that, for all $M > 1$ and infinitely many k, there is a set F of functions from a set X to a set Y of size k such that $\mathrm{opt}_{\mathrm{std}}(F) = M$ and

$$\mathrm{opt}_{\mathrm{bandit}}(F) \geq (1 - o(1))\,(|Y| \ln |Y|)\,\mathrm{opt}_{\mathrm{std}}(F), \tag{2}$$

and that an

$$\mathrm{opt}_{\mathrm{bandit}}(F) \leq (1 + o(1))\,(|Y| \ln |Y|)\,\mathrm{opt}_{\mathrm{std}}(F) \tag{3}$$

bound holds for all F.

Previous work. In addition to the bounds described above, on-line learning with bandit feedback, side-information and adversarially chosen examples has been heavily studied (see (Helmbold et al., 2000; Auer et al., 2002; Abe et al., 2003; Auer, 2002; Kakade et al., 2008; Chu et al., 2011; Bubeck and Cesa-Bianchi, 2012; Crammer and Gentile, 2013)). Daniely and Helbertal (2013) studied the price of bandit feedback in the agnostic on-line model, where the online learning algorithm is evaluated by comparison with the best mistake bound possible in hindsight obtained by repeatedly applying a classifier in F. The proof of (2) uses analytical tools that were previously used for experimental design (Rao, 1946, 1947), and hashing, derandomization and cryptography (Carter and Wegman, 1977; Luby and Wigderson, 2006). The proof of (3) uses tools based on the Weighted Majority algorithm (Littlestone and Warmuth, 1989; Auer and Long, 1999).

2. Preliminaries and main results

2.1 Definitions

Define $\mathrm{opt}_{\mathrm{bs}}(k, M)$ to be the best possible bound on $\mathrm{opt}_{\mathrm{bandit}}(F)$ in terms of $M = \mathrm{opt}_{\mathrm{std}}(F)$ and $k = |Y|$. In other words, $\mathrm{opt}_{\mathrm{bs}}(k, M)$ is the maximum, over sets X and sets F of functions from X to $\{0, ..., k-1\}$ such that $\mathrm{opt}_{\mathrm{std}}(F) = M$, of $\mathrm{opt}_{\mathrm{bandit}}(F)$.

We denote the limit supremum by $\overline{\lim}$.

2.2 Results

The following is our main result.

Theorem 1

$$\overline{\lim}_{M \to \infty} \overline{\lim}_{k \to \infty} \frac{\mathrm{opt}_{\mathrm{bs}}(k, M)}{kM \ln k} = 1.$$

2.3 The extremal case

For any prime p, let $F_L(p, n)$ be the set of all linear functions from $\{0, ..., p-1\}^n$ to $\{0, ..., p-1\}$, where operations are done with respect the finite field $GF(p)$.

In other words, for each $\mathbf{a} \in \{0, ..., p-1\}^n$, let $f_{\mathbf{a}} : \{0, ..., p-1\}^n \rightarrow \{0, ..., p-1\}$ be defined by

$$f_{\mathbf{a}}(\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x}) \mod p$$

and let $F_L(p, n) = \{f_{\mathbf{a}} : \mathbf{a} \in \{0, ..., p-1\}^n\}$.

The fact that

$$\text{opt}_{\text{std}}(F_L(p, n)) = n \tag{4}$$

for all primes $p \geq 2$ is essentially known (see (Shvaytser, 1988; Auer et al., 1995; Blum, 1998)). (An algorithm can achieve a mistake bound of n by exploiting the linearity of the target function to always predict correctly whenever $\mathbf{x}_t$ is in the span of previously seen examples. An adversary can force mistakes on any linearly independent set of the domain by answering whichever of 0 or 1 is different from the algorithm's prediction.)

3. Lower bounds

Our lower bound proof will use an adversary that maintains a *version space* (Mitchell, 1977), a subset of $F_L(p, n)$ that could still be the target. To keep the version space large no matter what the algorithm predicts, the adversary chooses a $\mathbf{x}_t$ for round t that divides it evenly. The first lemma analyzes its ability to do this.

Lemma 2 *For any $S \subseteq \{1, ..., p-1\}^n$, there is a $\mathbf{u}$ such that for all $z \in \{0, ..., p-1\}$,*

$$|\{\mathbf{s} \in S : \mathbf{s} \cdot \mathbf{u} = z \mod p\}| \leq |S|/p + 2\sqrt{|S|}.$$

Lemma 2 is similar to analyses of hashing (see (Blum, 2011)).

Lemma 2 is proved using the probabilistic method. The next two lemmas about the distribution of splits for random domain elements may already be known; see e.g. (Luby and Wigderson, 2006; Blum, 2011) for proofs of some closely related statements. We included proofs in appendices because we do not know a reference with proofs for exactly the statements needed here.

Lemma 3 *Assume $n \geq 1$. For $\mathbf{u}$ chosen uniformly at random from $\{0, ..., p-1\}^n$, for any $\mathbf{s} \in \{0, ..., p-1\}^n - \{\mathbf{0}\}$ for any $z \in \{0, ..., p-1\}$, we have*

$$\mathbf{Pr}(\mathbf{s} \cdot \mathbf{u} = z \mod p) = 1/p.$$

Proof: See Appendix A. ∎

Lemma 4 *Assume $n \geq 2$. For $\mathbf{u}$ chosen uniformly at random from $\{0, ..., p-1\}^n$, for any $\mathbf{s}, \mathbf{t} \in \{1, ..., p-1\}^n$ such that $\mathbf{s} \neq \mathbf{t}$, and for any $z \in \{0, ..., p-1\}$, we have*

$$\mathbf{Pr}(\mathbf{t} \cdot \mathbf{u} = z \mod p \mid \mathbf{s} \cdot \mathbf{u} = z \mod p) = 1/p.$$

Proof. See Appendix B. ∎

Armed with Lemmas 3 and 4, we are ready for the proof of Lemma 2.

Proof (of Lemma 2): Let S be an arbitrary subset of $\{1, ..., p-1\}^n$. Choose $\mathbf{u}$ uniformly at random from $\{0, ..., p-1\}^n$. For each $z \in \{0, ..., p-1\}$, let S_z be the (random) set of $\mathbf{s} \in S$ such that $\mathbf{s} \cdot \mathbf{u} = z \mod p$. Lemma 3 implies that, for all z,

$$\mathbf{E}(|S_z|) = |S|/p$$

and, since Lemmas 3 and 4 imply that the events that $\mathbf{s} \cdot \mathbf{u} = z$ are pairwise independent,

$$\mathbf{Var}(|S_z|) = \mathbf{Var}(1_{\mathbf{s} \cdot \mathbf{u} = z})|S| = (1/p)(1 - 1/p)|S| < |S|/p.$$

Using Chebyshev's inequality,

$$\mathbf{Pr}(|S_z| \geq |S|/p + 2\sqrt{|S|}) \ \leq \ \frac{1}{4p}.$$

Applying a union bound, with probability at least 3/4,

$$\forall z, |S_z| \leq |S|/p + 2\sqrt{S},$$

completing the proof. ∎

Now we are ready for the learning lower bound.

Lemma 5

$$\overline{\lim}_{n \to \infty} \overline{\lim}_{p \to \infty} \frac{\text{opt}_{\text{bandit}}(F_L(p,n))}{pn \ln p} \geq 1. \tag{5}$$

Proof: Choose $n \geq 3$ and $p \geq 5$. Consider an adversary that maintains a list F_t of members of

$$\{f_{\mathbf{a}} : \mathbf{a} \in \{1, ..., p-1\}^n\} \subseteq F_L(p,n)$$

that are consistent with its previous answers, always answers "no", and picks $\mathbf{x}_t$ for round t that splits F_t as evenly as possible; that is, $\mathbf{x}_t$ minimizes the maximum, over potential values of $\hat{y}_t$, of $|F_t \cap \{f : f(\mathbf{x}_t) = \hat{y}_t\}|$. As long as $|F_t| \geq p^2 \ln p$, Lemma 2 implies that,

$$\begin{aligned} |F_{t+1}| &\geq |F_t| - \frac{|F_t|}{p} - 2\sqrt{|F_t|} \\ &\geq |F_t| - \frac{|F_t|}{p} - \frac{2|F_t|}{p\sqrt{\ln p}} \\ &= \left(1 - \left(\frac{1 + 2/\sqrt{\ln p}}{p}\right)\right)|F_t|. \end{aligned}$$

Thus, by induction, we have

$$|F_t| \geq \left(1 - \left(\frac{1 + 2/\sqrt{\ln p}}{p}\right)\right)^{t-1} (p-1)^n.$$

The adversary can force m mistakes before $|F_t| < p^2 \ln p$ if

$$\left(1 - \frac{1 + 2/\sqrt{\ln p}}{p}\right)^{m-1} (p-1)^n \geq p^2 \ln p$$

which is true for $m = (1 - o(1))np \ln p$, proving (5). ∎

4. Upper bound

The upper bound proof closely follows the arguments in (Littlestone and Warmuth, 1989; Auer and Long, 1999).

Lemma 6 *For any set F of functions from some set X to $\{0, ..., k-1\}$,*

$$\text{opt}_{\text{bandit}}(F) \leq (1 + o(1))(k \ln k)\text{opt}_{\text{std}}(F).$$

Proof: Consider an algorithm A_b for the bandit model, which uses an algorithm A_s for the standard model as a subroutine, defined as follows. Algorithm A_b maintains a list of copies of algorithm A_s that have been given different inputs. For $\alpha = \frac{1}{k \ln k}$, each copy of A_s is given a weight: if it has made m mistakes, its weight is α^m. In each round, A_b uses these weights to make its prediction by taking a weighted vote over the predictions made by the copies of A_s.

Algorithm A_b starts with a single copy. Whenever it makes a mistake, all copies of A_s that made a prediction that was not used by A_b "forget" the round – their state is rewound as if the round did not happen. Each copy of A_s that voted for the winner is cloned, including its state, to make $k-1$ copies, and each copy is given a different "guess" of $f(x_t)$.

Let W_t be the total weight of all of the copies of A_s before round t. Since one copy of A_s always gets correct information, for all t, we have

$$W_t \geq \alpha^{\text{opt}_{\text{std}}(F)}. \tag{6}$$

On the other hand, after each round t in which A_b makes a mistake, copies of A_s whose total weight is at least W_t/k are cloned to make $k-1$ copies, each with weight $\alpha < 1/(k-1)$ times its old weight. Thus

$$W_{t+1} \leq (1 - 1/k)W_t + (1/k)(\alpha(k-1)W_t) < (1 - 1/k)W_t + \alpha W_t$$

and, after A_b has made m mistakes,

$$W_t < (1 - 1/k + \alpha)^m < e^{-(1/k-\alpha)m}.$$

Combining with (6) yields

$$e^{-(1/k-\alpha)m} > \alpha^{\text{opt}_{\text{std}}(F)}$$

which implies $m \leq \frac{\ln(1/\alpha)\text{opt}_{\text{std}}(F)}{1/k-\alpha}$ and substituting the value of α completes the proof. ∎

5. Putting it together

Theorem 1 follows from (4), Lemma 5, and Lemma 6.

6. Two open problems

There appears to be a $\Theta(\sqrt{\log |Y|})$ gap between the best known upper and lower bounds on the cost of bandit feedback for on-line multiclass learning in the agnostic model (Daniely and Helbertal, 2013). Can the analysis of $F_L(p, n)$ play a role in closing this gap?

It is not hard to see that $\text{opt}_{\text{bs}}(k, 1) = k - 1 = \Theta(k)$, and the proof of Lemma 5 implies that $\text{opt}_{\text{bs}}(k, 3) = \Theta(k \log k)$. What about $\text{opt}_{\text{bs}}(k, 2)$?

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Appendix A. Proof of Lemma 3

Pick i such that $s_i \neq 0$. We have

$$\begin{aligned}\mathbf{Pr}(\mathbf{u}\cdot\mathbf{s} = z \mod p) &= \mathbf{Pr}(u_i s_i = z - \sum_{j\neq i} u_j s_j \mod p) \\ &= \mathbf{Pr}(u_i = \left(z - \sum_{j\neq i} u_j s_j\right) s_i^{-1} \mod p) \\ &= 1/p,\end{aligned}$$

completing the proof.

Appendix B. Proof of Lemma 4

Let i be one component such that $s_i \neq t_i$. Let $\mathbf{s}'$, $\mathbf{t}'$ and $\mathbf{u}'$ be the projections of $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{u}$ onto the indices other than i.

Lemma 3 implies that $\mathbf{s}'\cdot\mathbf{u}' \mod p$ is distributed uniformly on $\{0, ..., p-1\}$. Thus, after conditioning on the event that $\mathbf{s}\cdot\mathbf{u} = z \mod p$, u_i is uniform over $\{0, ..., p-1\}$, which implies

$$\begin{aligned}
&\mathbf{Pr}(\mathbf{t} \cdot \mathbf{u} = z \mod p \mid \mathbf{s} \cdot \mathbf{u} = z \mod p) \\
&= \mathbf{Pr}(u_i(t_i - s_i) = (\mathbf{s}' - \mathbf{t}') \cdot \mathbf{u}' \mod p \mid \mathbf{s} \cdot \mathbf{u} = z \mod p) \\
&= 1/p,
\end{aligned}$$

completing the proof.